Backpaper - Computer Science 2 (2021-22)
Time: 3 hours.
Attempt all questions, giving proper explanations.

1. How is the number -50.875 stored as a floating point number in the computer ? Give the sign, mantissa and exponent. **[6 marks]**

2. Consider the solution to $x = \log(3x+1)$ in $[1,3]$. Consider the iterations $x_{k+1} = \log(3x_k+1)$. Starting from $x_0 = \frac{3}{2}$ how many iterations are necessary before we are within 10^{-6} of the solution? **[6 marks]**

3. Consider solving the equation $f(x) = 0$. Describe Newton's method, Secant method and Bisection method. **[6 marks]**

4. Use Gaussian elimination to solve

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 6 \end{bmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \textbf{[6 marks]}$$

5. Use Gram-Schmidt orthogonalization process to find an orthonormal basis for the span of the following vectors in $\mathbf{R}^4$:

$$\Big\{(1,2,1,1),\ (1,0,2,0),\ (1,3,1,1)\Big\}. \qquad \textbf{[6 marks]}$$

6. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Apply the first iteration of the *classical Jacobi method*: Find the orthogonal matrix $\mathbf{P}$ such that the $(3,4)$th entry of $\mathbf{A}^{(1)} := \mathbf{P}^T\mathbf{A}\mathbf{P}$ is 0. **[6 marks]**

7. Consider an infinitely differentiable function $f : [0,1] \to \mathbf{R}$.
   (a) Write down the Newton-Cotes formula for $\int_0^1 f(x)dx$ with 4 equally spaced points $0 = x_0 < x_1 < x_2 < x_3 = 1$. **[4 marks]**
   (b) What is the error in approximating the integral by the approximation? **[2 marks]**

8. Let $f : [0,T] \times \mathbf{R} \to \mathbf{R}$ be such that
   - f is continuous on $[0,T] \times \mathbf{R}$,
   - $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial x}$ are bounded on $[0,T] \times \mathbf{R}$.

   Consider the solution $x : [0,T] \to \mathbf{R}$ of the differential equation

$$\frac{dx}{dt} = f(t,x),$$
$$x(0) = \alpha.$$

   Split the interval $[0,T]$ into subintervals of size $h > 0$ so that $0 = t_0 < t_1 < \cdots < t_n = T$ with $t_i = ih$ are the grid points. Consider the Euler approximation of x on the grid points :

$$\tilde{x}(t_i) = \tilde{x}(t_{i-1}) + f\left(t_{i-1}, \tilde{x}(t_{i-1})\right)h, \qquad 1 \le i \le n$$
$$\tilde{x}(0) = \alpha.$$

   Prove in **complete detail** that $\sup_i |x(t_i) - \tilde{x}(t_i)| = O(h)$ as $h \to 0$. **[8 marks]**